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Mixed rational–soliton solutions of two differential-difference equations in Casorati determinant form

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Abstract

By reconsidering soliton solutions of the Toda lattice and differential-difference KdV equation in the Casorati determinant form with new entries, we obtain rational and mixed rational–soliton solutions in the Casorati determinant form. All these solutions are verified by direct substitutions into bilinear equations. The method used is general and can apply to other discrete systems.

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1. Introduction

Since GGKM's discovery [1] of the inverse scattering transform and Hirota's discovery [2] of the bilinear method, many studies have been devoted to finding soliton solutions for the evolution equations. Hirota [3] also proposed a kind of Bäcklund transformation (BT) in a bilinear form, by which solutions can be derived recursively and easily. A soliton solution can also be represented in terms of a Wronskian [4, 5]. By using bilinear equations (or bilinear BT) and compact forms of derivatives of a Wronskian, Freeman and Nimmo [6–9] developed a procedure, which we call the Wronskian technique, to verify solution by direct substitutions.

Rational solution of the KdV equation was first investigated by Moser and co-workers [10, 11]. In 1978, Ablowitz and Satsuma [12] developed a simple method to find rational solutions by taking the long wave limit in the multisoliton solution in Hirota's form. This method is general and has been used for some differential-difference systems [13, 14]. Another different approach was proposed by Hu and Clarkson [15] in 1995. By using Hirota's bilinear formalism and BTs, they gave the nonlinear superposition formulae of rational solutions for three differential-difference equations. It should be noted that following thoughts of Ablowitz and Satsuma [12], Nimmo and Freeman [16] verified rational solutions of the KdV equation in the Wronskian form. However, they stated at that time that [16]: 'It might be hoped that the rational solutions of other equations of the type whose solutions take Wronskian form may be

obtained in a similar way, however we have found this not to be possible.' In 1992, Matveev [17] presented a new derivation of rational solution for the KdV equation by generalizing the Wronskian formula. The Wronskian which he considered can be written as a product of an exponential function and a determinant with entries of polynomials of x and t . As the exponential function disappears when we recover a solution, the determinant can denote a rational solution.

Mixed rational–soliton solutions (quasi-soliton) of the KdV equation were discovered by Ablowitz and Cornille [18]. In 1996, Cârstea and Grecu [13] obtained 1-rational multisoliton solutions for the Toda lattice using bilinear BT. They also gave the Wronskian of 1-rational N -soliton solutions. Recently, this kind of mixed solution for some other differential-difference equations was investigated by Narita [19–21]. He developed some new representations and new procedures for mixed solutions. The generalized Wronskian presented by Matveev [17] can also denote a sort of mixed solution which he calls a positon–soliton solution [17, 22].

It is well known that the solution in Wronskian form not only allows direct verifications, but is also easy to calculate. In this paper, for two differential-difference equations, we consider their rational and mixed solutions in the form of the discrete analogue of a Wronskian, the Casorati determinant (CD), namely, a 'Wronskian' constructed in terms of the shift of discrete variable n . The key is to choose new suitable entries in the CD, though it does not lead to new soliton solutions. However, such a new choice allows us to derive rational solutions in the CD form following the method proposed by Nimmo and Freeman [16]. And further, we develop Nimmo–Freeman's procedure to obtain arbitrarily mixed multirational multisoliton solutions in the CD form. All these solutions are verified by direct substitutions into bilinear equations. Two examples considered in this paper are the Toda lattice and differential-difference KdV equation. The method used here is general and can apply to other discrete systems.

The paper is organized as follows. In section 2, soliton, rational and mixed solutions in the CD form for the Toda lattice are obtained. In section 3, we obtain similar results for the differential-difference KdV equation. Finally, a conclusion is given.

2. Rational and mixed solutions in the CD form of the Toda lattice

In this section, we derive rational and mixed multirational–multisoliton solutions in the CD form for the Toda lattice, which is the most known and studied nonlinear integrable lattice model.

2.1. Soliton solution of the Toda lattice

We first recall some results about the Toda lattice. The Toda lattice is [23]

$$x_{n,tt} = e^{x_{n-1}-x_n} - e^{x_n-x_{n+1}} \quad (2.1)$$

with bilinear form [24]

$$[D_t^2 - 2(\cosh D_n - 1)] f_n \cdot f_n = 0 \quad (2.2)$$

or

$$f_n f_{n,tt} - f_{n,t}^2 - f_{n-1} f_{n+1} + f_n^2 = 0. \quad (2.3)$$

Here D is the well-known Hirota bilinear operator

$$D_t^m f \cdot g = (\partial_t - \partial_{t'})^m f(n, t) g(n, t')|_{t'=t} \quad e^{D_n} a_n \cdot b_n = a_{n+1} b_{n-1}$$

and the transformation is

$$y_n = x_n - x_{n-1} \quad e^{-y_n} - 1 = (\ln f_n)_{tt}. \quad (2.4)$$

In [9], Nimmo constructed a Wronskian

$$f_n = \begin{vmatrix} \psi_1(n, t) & \psi_1^{(1)}(n, t) & \cdots & \psi_1^{(N-1)}(n, t) \\ \psi_2(n, t) & \psi_2^{(1)}(n, t) & \cdots & \psi_2^{(N-1)}(n, t) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_N(n, t) & \psi_N^{(1)}(n, t) & \cdots & \psi_N^{(N-1)}(n, t) \end{vmatrix} = |0, 1, \dots, N-1| = |\widehat{N-1}| \quad (2.5)$$

where

$$\psi_j(n, t) = a_j^+ e^{k_j n + t} + a_j^- e^{-k_j n + t} \quad a_j^+, a_j^-, k_j \in R \quad (j = 1, 2, \dots, N) \quad (2.6)$$

and $\psi_j^{(l)}(n, t) = d^l \psi_j(n, t) / dt^l$. Here we follow Nimmo's notation [9]; let $\widehat{N-j}$ indicate the set of consecutive columns $0, 1, 2, \dots, N-j$ and let $\widetilde{N-j}$ indicate the set of consecutive columns $1, 2, \dots, N-j$. Noting that $\psi_j(n, t)$ satisfies

$$2 \cosh k_j \psi_j(n, t) = \psi_j(n-1, t) + \psi_j(n+1, t) \quad k_j \in R \quad (2.7a)$$

$$\psi_{j,t}(n, t) = \psi_j(n+1, t) \quad (2.7b)$$

and with the help of formulae

$$\left(2 \sum_{j=1}^N \cosh k_j \right) |\widehat{N-1}| = |\widehat{N-2}, N| + |-\widetilde{1}, \widetilde{N-1}| \quad (2.8)$$

$$\begin{aligned} & \left(2 \sum_{j=1}^N \cosh k_j \right) |\widehat{N-2}, N| \\ &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1| + |\widehat{N-1}| + |-\widetilde{1}, \widetilde{N-2}, N| \end{aligned} \quad (2.9)$$

Nimmo [9] proved that equation (2.5) with entries (2.6) solved the bilinear equation (2.3).

In fact, similar to Nimmo's proof [9], it is easy to show that for any $\{\psi_j(n, t)\}$ which enjoy equations (2.7a) and (2.7b), Wronskian (2.5) solves (2.3), i.e. equation (2.6) is a particular choice. On the other hand, under the condition of (2.7b), equation (2.5) is just a CD.

In this paper, we define

$$\phi_j(n, t) = a_j^+ e^{k_j n + t(e^{k_j} - 1)} + a_j^- e^{-k_j n + t(e^{-k_j} - 1)} = e^{-t} \psi_j(n, t) \quad (2.10)$$

and construct a CD in terms of $\phi_j(n, t)$, i.e.

$$F_n = \begin{vmatrix} \phi_1(n, t) & \phi_1(n+1, t) & \cdots & \phi_1(n+N-1, t) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_N(n, t) & \phi_N(n+1, t) & \cdots & \phi_N(n+N-1, t) \end{vmatrix} = |0, 1, \dots, N-1| = |\widehat{N-1}|. \quad (2.11)$$

Here we still adopt the notation ' $\widehat{}$ ' and ' $\widetilde{}$ '. Such a CD still satisfies the identities (2.8) and (2.9) but has special derivatives with respect to t , for example, $F_{n,t} = -N F_n + |\widehat{N-2}, N|$. In a similar way to [9], it is not difficult to verify that F_n solves equation (2.3) by direct substitution. An alternative verification comes from the facts $F_n = e^{-Nt} f_n$ and

$$F_n F_{n,t} - F_{n,t}^2 - F_{n-1} F_{n+1} + F_n^2 = e^{-2Nt} (f_n f_{n,t} - f_{n,t}^2 - f_{n-1} f_{n+1} + f_n^2).$$

Obviously, F_n and f_n recover the same solution from the transformation (2.4). However, such a CD allows us to derive rational solutions following the method proposed by Nimmo and Freeman [16].

2.2. Rational solution

We first expand $e^{k_j n + t(e^{k_j} - 1)}$ as

$$e^{k_j n + t(e^{k_j} - 1)} = \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{i!} n^i \beta_s(t) k_j^{i+s} = \sum_{m=0}^{\infty} \left[\sum_{s=0}^{\infty} \frac{n^{m-s}}{(m-s)!} \beta_s(t) \right] k_j^m \tag{2.12}$$

where $\beta_s(t)$, a function of t , is determined by

$$\sum_{s=0}^{\infty} \beta_s(t) k_j^s = \sum_{l=0}^{\infty} \frac{t^l}{l!} \left(\sum_{h=1}^{\infty} \frac{1}{h!} k_j^h \right)^l \tag{2.13}$$

or, equivalently, by the following explicit formulae:

$$\beta_0(t) = 1 \quad \beta_1(t) = t \quad \beta_2(t) = \frac{1}{2}(t + t^2) \quad \beta_3(t) = \frac{1}{6}t + \frac{1}{2}t^2 + \frac{1}{6}t^3$$

$$\begin{aligned} \beta_s(t) = & t \frac{1}{s!} + \frac{1}{2!} t^2 \sum_{h_1+h_2=s} \frac{1}{h_1!} \frac{1}{h_2!} + \frac{1}{3!} t^3 \sum_{h_1+h_2+h_3=s} \frac{1}{h_1!} \frac{1}{h_2!} \frac{1}{h_3!} + \dots \\ & + \frac{1}{(s-2)!} t^{s-2} \sum_{h_1+h_2+\dots+h_{s-2}=s} \left(\prod_{j=1}^{s-2} \frac{1}{h_j!} \right) \\ & + \frac{1}{2} \frac{1}{(s-2)!} t^{s-1} + \frac{1}{s!} t^s \quad (h_j \geq 1). \end{aligned} \tag{2.14}$$

We also have

$$e^{-k_j n + t(e^{-k_j} - 1)} = \sum_{m=0}^{\infty} \left[\sum_{s=0}^{\infty} \frac{n^{m-s}}{(m-s)!} \beta_s(t) \right] (-k_j)^m. \tag{2.15}$$

Now we consider the rational solution when $a_j^+ = a_j^- = 1$ (or, generally, $a_j^+/a_j^- = 1$) in (2.10). In this case, we have $\phi_j(n, t) = \sum_{i=0}^{\infty} a_i(n, t) k_j^{2i}$ and

$$a_i(n, t) = \sum_{s=0}^{2i} \frac{n^{2i-s}}{(2i-s)!} \beta_s(t). \tag{2.16}$$

Taking $(k_1, k_2, \dots, k_N) \rightarrow (0, 0, \dots, 0)$ yields

$$F_n \sim |\widehat{N-1}|_R \mathcal{K} \tag{2.17}$$

where

$$|\widehat{N-1}|_R = \begin{vmatrix} a_0(n, t) & a_0(n+1, t) & \dots & a_0(n+N-1, t) \\ \vdots & \vdots & \dots & \vdots \\ a_{N-1}(n, t) & a_{N-1}(n+1, t) & \dots & a_{N-1}(n+N-1, t) \end{vmatrix} \tag{2.18}$$

and \mathcal{K} is a Vandermonde determinant

$$\mathcal{K} = \begin{vmatrix} 1 & k_1^2 & \dots & k_1^{2(N-1)} \\ \vdots & \vdots & \dots & \vdots \\ 1 & k_N^2 & \dots & k_N^{2(N-1)} \end{vmatrix} = \prod_{1 \leq i < j \leq N} (k_j^2 - k_i^2).$$

Noting some identities about $|\widehat{N-1}|_R$ derived from (2.8) and (2.9):

$$2N|\widehat{N-1}|_R = |\widehat{N-2}, N|_R + |-1, \widetilde{N-1}|_R$$

$$2N|\widehat{N-2}, N|_R = |\widehat{N-3}, N-1, N|_R + |\widehat{N-2}, N+1|_R + |\widehat{N-1}|_R + |-1, \widetilde{N-2}, N|_R$$

we can verify $|\widehat{N-1}|_R$ to be the solution of the bilinear equation (2.3). That is to say, the rational solution of the Toda lattice can be generated from $|\widehat{N-1}|_R$ in which $a_i(n, t)$ is determined from equation (2.16).

In the cases of $a_j^+ = -a_j^- = 1$ and $|a_j^+| \neq |a_j^-|$, the rational solutions can also be denoted by (2.18), but $a_i(n, t)$ is defined respectively by

$$a_i(n, t) = \sum_{s=0}^{2i+1} \frac{n^{2i+1-s}}{(2i+1-s)!} \beta_s(t) \tag{2.19}$$

and

$$a_i(n, t) = \sum_{s=0}^i \frac{n^{i-s}}{(i-s)!} \beta_s(t). \tag{2.20}$$

2.3. Mixed $(N - l)$ -rational l -soliton solution

We still consider the case of $a_j^+ = a_j^- = 1$. Without losing generality, we expand F_n with respect to the first l rows,

$$F_n = \sum_{0 \leq i_1 < i_2 < \dots < i_l \leq N-1} (-1)^{\sum_{j=1}^l (j+i_j+1)} \cdot A(i_1, i_2, \dots, i_l) \cdot B(i_1, i_2, \dots, i_l) \tag{2.21}$$

$(1 \leq l \leq N - 1)$

where

$$A(i_1, i_2, \dots, i_l) = \begin{vmatrix} \phi_1(n+i_1, t) & \dots & \phi_1(n+i_l, t) \\ \vdots & \dots & \vdots \\ \phi_l(n+i_1, t) & \dots & \phi_l(n+i_l, t) \end{vmatrix}$$

and $B(i_1, i_2, \dots, i_l)$ is the cofactor of $A(i_1, i_2, \dots, i_l)$.

If we only take the limit of $(k_{l+1}, k_{l+2}, \dots, k_N) \rightarrow (0, 0, \dots, 0)$ in equation (2.21), we have

$$F_n \sim |\widehat{N-1}|_M \mathcal{K} \tag{2.22}$$

where

$$|\widehat{N-1}|_M = \begin{vmatrix} \phi_1(n, t) & \phi_1(n+1, t) & \dots & \phi_1(n+N-1, t) \\ \vdots & \vdots & \dots & \vdots \\ \phi_l(n, t) & \phi_l(n+1, t) & \dots & \phi_l(n+N-1, t) \\ a_0(n, t) & a_0(n+1, t) & \dots & a_0(n+N-1, t) \\ \vdots & \vdots & \dots & \vdots \\ a_{N-l-1}(n, t) & a_{N-l-1}(n+1, t) & \dots & a_{N-l-1}(n+N-1, t) \end{vmatrix} \tag{2.23}$$

$$\mathcal{K} = \begin{vmatrix} 1 & k_{l+1}^2 & \cdots & k_{l+1}^{2(N-l-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & k_N^2 & \cdots & k_N^{2(N-l-1)} \end{vmatrix} = \prod_{l+1 \leq i < j \leq N} (k_j^2 - k_i^2).$$

We can finish the verification of the solution with the help of the formulae

$$\begin{aligned} \left[2(N-l) + 2 \sum_{j=1}^l \cosh k_j \right] |\widehat{N-1}|_M &= |\widehat{N-2}, N|_M + |-\widetilde{1}, \widetilde{N-1}|_M \\ \left[2(N-l) + 2 \sum_{j=1}^l \cosh k_j \right] |\widehat{N-2}, N|_M &= |\widehat{N-3}, N-1, N|_M + |\widehat{N-2}, N+1|_M + |\widehat{N-1}|_M + |-\widetilde{1}, \widetilde{N-2}, N|_M. \end{aligned}$$

Thus, $|\widehat{N-1}|_M$ indicates a mixed $(N-l)$ -rational l -soliton solution in the CD form for the Toda lattice. For the case of $a_j^+ = a_j^- = 1$, $a_i(n, t)$ is defined by (2.16), for the cases of $a_j^+ = -a_j^- = 1$ and $|a_j^+| \neq |a_j^-|$, $a_i(n, t)$ is defined respectively by (2.19) and (2.20). These mixed solutions are more general than the results in [13] and differ from [25] which considered positon-soliton solutions through the Darboux transformation.

3. Solutions of the differential-difference KdV equation

The differential-difference KdV equation under consideration is [16, 26]

$$-\left(\frac{W_n}{1+W_n} \right)_t = W_{n-\frac{1}{2}} - W_{n+\frac{1}{2}}. \tag{3.1}$$

It can be written in the bilinear form [27]

$$\sinh\left(\frac{1}{4}D_n\right) [D_t - 2 \sinh\left(\frac{1}{2}D_n\right)] f_n \cdot f_n = 0 \tag{3.2}$$

with the transformation

$$W_n = \frac{(\cosh \frac{1}{2}D_n) f_n \cdot f_n}{f_n^2} - 1. \tag{3.3}$$

We can obtain soliton, rational and mixed solutions of this equation by employing similar procedures to section 2 above. Here, we only list the main results.

Soliton solution can be denoted by (2.11), where $\phi_j(n, t)$ is given by (2.10). Rational and mixed solutions are denoted respectively by (2.18) and (2.23), where for the case of $a_j^+ = a_j^- = 1$, $a_i(n, t)$ is defined by (2.16) and for the cases of $a_j^+ = -a_j^- = 1$ and $|a_j^+| \neq |a_j^-|$, $a_i(n, t)$ is defined respectively by (2.19) and (2.20). Obviously, these results are the same as the Toda lattice, but the solutions of equation (3.1) are recovered from (3.2).

4. Conclusion

We have reconsidered soliton solutions in the CD form for the Toda lattice and differential-difference KdV equation by re-choosing entries. Such a choice allows us to derive rational solutions in the CD form following the method proposed by Nimmo and Freeman [16]. We also further develop their procedure to obtain mixed $(N-l)$ -rational l -soliton solutions. All these solutions are verified by direct substitutions into bilinear equations. The method used in this paper is general and can apply to other discrete systems.

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